Decoherence of a Spin Qubit Coupled with a Spin Chain

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In this report, we examine the relaxation phenomena of a spin qubit coupled to a spin chain with a $1/r^2$ interaction.

§1. Introduction

Among various proposals for quantum computations, quantum bits (qubits) in solid state materials, such as superconducting Josephson junctions,[1] and quantum dots,[2)–4] have the advantage of scalability. Such coherent two level systems constitute qubits and a quantum computation can be carried out as a unitary operation appied to multiple qubit systems. It is essential that this quantum coherencre be maintained during computation. However, dephasing is difficult to avoid, due to the system's interaction with the environment. The decay of off-diagonal elements of the qubit density matrix signals the occurrence of dephasing. Various environments can cause dephasing. In solid state systems, the effect of phonons is ubiquitous.[5] The effect of electromagnetic fluctuation has been extensively studied for Josephson junction charge qubits.[6]

The fluctuations of the nuclear spins of impurities can also be a cause of dephasing. It has recently been shown experimentally that the coupling between the spin of an electron in a quantum dot and the environment is very weak.[7),8),10),11] For this reason, the dephasing time of a spin qubit conjectured to be very long. However, both donor impurities and nuclear spins in semiconductors[9] have been suggested as possible building blocks for feasible quantum dots architectures. The proposal based on experimental findings of quantum computation using an Si^{29} array is another possibility.[9] For a spin qubit system in 2D, the coupling between nearest neighbor spin chain can cause a dephasing. When an interaction is between the qubits themselves, it can in principle be incorporated into the quantum computer Hamiltonian, although this would lead to more complicated gate sequences. Therefore it is instructive to analyze the error introduced by ignoring some of these interactions, as done in the case of dipolar coupled spin qubits.[12] For a quantum spin chain with a $1/r^2$ interaction, an exact expression of the dynamical correlation function has been obtained.[14] Using this expression, we consider the case in which one qubit is coupled to each spin of the spin chain. We examine the relaxation phenomena of a spin qubit in array that is coupled to a spin chain with long-range interactions. Integrating over the spin chain variables, we obtain the influence function of the qubit system. Using this influence function, we examine the dephasing of the qubit's

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density matrix. In the present study, we especially concentrated on the correlation effect between the spin of the qubit and that of the spin chain, and show how the characteristic nature of environment appears. The result is that the dephsing rate increases as a function of the intrachain interactions, because the fluctuations of the spinon is suppressesd interaction increases.

§2. Hamiltonian

We examine the Hamiltonian which is given by,

$$\mathcal{H} = H_{qb} + H_{qb-\mathrm{spin}} + H_{\mathrm{spin}} \tag{1}$$
$$H_{qb} = \hbar \mathit{\Delta} I_z, \tag{2}$$
$$H_{\mathrm{int}} = \gamma_N \hbar^2 (A_{zz} I_z S_{z0} + A_{\perp}(I_+ S_{-0} + I_- S_{+0})), \tag{3}$$
$$H_{\mathrm{spin}} = J \sum_{i,j=N}^{N} [d(x_i - x_j)]^2 \mathbf{S}_i \cdot \mathbf{S}_j. \tag{4}$$

where N is the number of sites, and $d(n) = (N/\pi)\sin(\pi n/N) \to n$ as $N \to \infty$. Also i is the index of N lattice. $\mathit{\Delta}$ is the effective magnetic field applied to a spin qubit, and $A_{zz}\gamma_N \hbar$ is the magnitude of the fluctuations in the coupling constant. A system of this kind has also been examined in the context of a giant spin that is coupled to a surrounding nuclear spin interacting bilinearly with a spin bath in a magnetic field and with internal interactions of canonical form.[15),16)] In this study we examine the dephasing behavior by using an exact dynamical correlation function and examine the characteristic nature of a bath that interacts with itself.

§3. Influence function

We examine first a pure dephasing event ($A_{\perp} = 0$), and then we study dephasing with a dissipation event ($A_{\perp} \neq 0$). The density matrix of spin qubits is given by

$$\rho(I^f_{z+}, I^f_{z-}) = \int_{I_{z+}(0)=I^i_{z+}, I_{z-}(0)=I^i_{z-}}^{I_{z+}(t)=I^f_{z+}, I_{z-}(t)=I^f_{z-}} [dI_{z+}][dI_{z-}] \exp(\frac{i}{\hbar}(I_{qb}[I_{z-}] - I_{qb}[I_{z+}])) F[I_{z+}, I_{z-}], \tag{5}$$

where $I_{qb}[I_{\pm}] = \int_0^t \hbar \mathit{\Delta} I_{z\pm}$, and the influence function (IF) is defined as

$$F[I_{z+}, I_{z-}] = \int [d\mathbf{S}_{+i}][d\mathbf{S}_{-i}] \delta(\mathbf{S}_{+i}(t) - \mathbf{S}_{-i}(t)) \rho(\mathbf{S}_i(0), \mathbf{S}_j(0)) \exp\{\frac{i}{\hbar}(I[S_+] - I[S_-])\}. \tag{6}$$

For the following investigation, we assume $A_{\perp}$=0 and that the environment system action is given by $I[\mathbf{S}] = I_0[\mathbf{S}] + I_{int}[I_z, S^0_z]$, with free part

$$I_0[S] = \frac{(A_{zz}\gamma_N \hbar^2)^2}{4\hbar^2} \int_0^t \int_0^{t_1} dt_1 dt_2 \sum_{i.j} \mathbf{S}_i(t_1) \mathbf{\Delta}^{-1}_{00p}(t_1, i, t_2, j) \mathbf{S}_j(t_2)$$
$$+ i\frac{\theta}{4\pi} \int dx \int_0^t dt \mathbf{S} \cdot (\frac{\partial \mathbf{S}}{\partial x} \times \frac{\partial \mathbf{S}}{\partial t}) \tag{7}$$

where $\theta = 2\pi n$ and $\mathbf{\Delta}_{00p}(t_1, i, t_2, j)$ is the free propagator of the environmental system at zero temperature, which is defined on a closed time path and has four components:[17)]

$$\mathbf{\Delta}_{00p}(i, t_1, j, t_2) = \begin{pmatrix} \mathbf{\Delta}_{00}^{++}(i, t_1, j, t_2) & \mathbf{\Delta}_{00}^{+-}(i, t_1, j, t_2) \\ \mathbf{\Delta}_{00}^{-+}(i, t_1, j, t_2) & \mathbf{\Delta}_{00}^{--}(i, t_1, j, t_2) \end{pmatrix}. \tag{8}$$

When $A_\perp = 0$, $\mathbf{\Delta}_{00p} \to \mathit{\Delta}_{00p}^z$, only the z component of the qubit is important. For the Berry phase term, because n is odd, the environment is a one-dimensional spin-1/2 system.[18)] In the present model, the spin chain is a gapless solvable model (a special point in spin chain systems). Therefore, we can completely integrate over the spin chain degree of freedom and the Berry phase term does not make system to be distempered. Thus, introducing the incoming interaction picture for the environment, and integrating out the spin chain environment, we can easily verify that Eq. (8) becomes,[17)]

$$\begin{aligned} F[I_{z+}, I_{z-}] = \exp[-i\frac{(A_{zz}\gamma_N\hbar^2)^2}{4\hbar^2} &\int_0^t \int_0^{t_1} dt_1 dt_2 \\ &(I_{z+}(t_1)\mathit{\Delta}_{00}^{z++}(0, t_1, 0, t_2)I_{z+}(t_2) + I_{z-}(t_1)\mathit{\Delta}_{00}^{z--}(0, t_1, 0, t_2)I_{z-}(t_2) \\ &- I_{z+}(t_1)\mathit{\Delta}_{00}^{z+-}(0, t_1, 0, t_2)I_{z-}(t_2) - I_{z-}(t_1)\mathit{\Delta}_{00}^{z-+}(0, t_1, 0, t_2)I_{z+}(t_2))]. \end{aligned} \tag{9}$$

For convenience we change the coordinates, $\eta \equiv (I_{z+} + I_{z-})/2, \xi \equiv (I_{z+} - I_{z-})/2$. The coordinates of η and ξ, given by η and ξ are called the "sojourn" and "blip". In terms of these variables, the density matrix is described by $\rho(\eta = 1) = |\uparrow\rangle\langle\uparrow|, \rho(\eta = -1) = |\downarrow\rangle\langle\downarrow|, \rho(\xi = 1) = |\uparrow\rangle\langle\downarrow|, \rho(\xi = -1) = |\downarrow\rangle\langle\uparrow|$.

Then, the IF becomes

$$\begin{aligned} F[\eta, \xi] = \exp[-i\frac{(A_{zz}\gamma_N\hbar)^2}{4} &\int_0^t \int_0^{t_1} dt_1 dt_2 \{\xi(t_1)G^R(t_1, 0, t_2, 0)\eta(t_2) \\ &+ \eta(t_1)G^A(t_1, 0, t_2, 0)\xi(t_2) - \xi(t_1)G^K(t_1, 0, t_2, 0)\xi(t_2)\}], \end{aligned} \tag{10}$$

where $G^R(t_1, 0, t_2, 0)$, $G^A(t_1, 0, t_2, 0)$ and the $G^K(t_1, 0, t_2, 0)$ are the retarded Green function, advanced Green function and Keldysh Green function for $i = j = 0$. It should be noted that the starting Hamiltonians are those given Eqs.(1) $\sim$ (4).[17)] Because the propagator of a qubit has no time dependence, the blip state and sojourn state do not change in time. Therefore, when we choose the initial condition of the qubit density matrix to be a coherent state, such as $\rho(t = 0) = \pm(|\uparrow\rangle\langle\downarrow| \pm |\downarrow\rangle\langle\uparrow|)$, time evolution occurs only in the off-diagonal channel. In addition, even if we start from an off-diagonal state, the interaction Hamiltonian does not allow a spin flip process. Hence the blip state does not change in time. Therefore we can set $\xi(t) = \xi(= \pm 1)$ and $\eta(t) = 0$ for all t. This leads to the following exact expression of the dephasing rate :

$$\rho(\xi, t) = \exp[i\mathit{\Delta}t - i\frac{(A_{zz}\gamma_N\hbar)^2}{4} \int_0^t \int_0^{t_1} dt_1 dt_2 G^K(t_1, 0, t_2, 0)]$$

$$= \exp[i\varDelta t - \frac{(A_{zz}\gamma_N\hbar)^2}{4}\int_0^t\int_0^{t_1} dt_1 dt_2 \{S_0^z(t_1), S_0^z(t_2)\}]. \qquad (11)$$

Here, $\{S_0^z, S_0^z\}$ is the symmetrized correlation function, defined by $\{A,B\} = AB + BA$, because the Keldysh Green function is a symmetrized correlation function for the base of the present Hilbert space.[18)]

§4. Results

First, we examine a pure dephasing event. (i.e., $A_\perp$=0) Then using analytical expressions, the time evolution of the off-diagonal density matrix is given by

$$-\ln\rho(\xi = \pm 1, t) = i\varDelta t + \frac{(A_{zz}\gamma_N\hbar^2)^2}{4\hbar^2}\int_0^t\int_0^{t_1} dt_1 dt_2 \{S_0^z(t_1), S_0^z(t_2)\}.$$

Here, $E = E(\lambda_1.\lambda_2) = \frac{\pi v}{2}(\lambda_1^2 + \lambda_2^2 - 2\lambda_1^2\lambda_2^2)$, where v is the spinon velocity. We carried out the numerical estimation at zero temperature.

In the initial regime, $\frac{\pi v t}{2} \ll 1$, we have $-\mathrm{Re}\ln\rho(t) = \frac{(A_{zz}\gamma_N\hbar)^2}{8}t^2$. This is Gaussian behavior which is displayed in the initial regime. This comes the memory effect of the spin chain reservoir. In the stationary regime, we have $-\mathrm{Re}\ln\rho(t) = \frac{(A_{zz}\gamma_N\hbar)^2}{8v}t$. This behavior leads to exponential decay of the off-diagonal element of the density matrix in the long time tail. Therefore, after tracing out the spin chain system, the qubit density matrix exhibits a maximal mixed state, i.e., an entangled state is formed. In the intermediate regime ($1 < \frac{\pi v t}{2} < 50$) , the dephasing rate exibits oscillatory behavior. Due to the restriction of the spin magnitude of the spin chain, during the transient regime between decoupled state and an entangled state there exists oscillatory behavior. In the stationary limit ($50 \ll \frac{\pi v t}{2}$), the maximally entangled state is formed. Therefore, after we trace out the spin chain system, shows the qubit density matrix is in the maximally mixed state. The time traces qubit and an spin chain are separable state to maximally entangled state .

The disappearance of the diagonal elements of the qubit indicate this behavior. In another context this behavior represents an effect of the qubit system on the environment.

We now examine the finite temperature behavior. In the frequency regime, we carry out the Fourier transform of the integral in Eq. (12). Then, the expression for the of dephasing rate is

$$-\mathrm{Re}\ln\rho(t) = \frac{(A_{zz}\gamma_N\hbar)^2}{4}\int_{-\infty}^{\infty} d\omega\rangle\{S_z(0,\omega), S_z(0,-\omega)\}\langle\left(\frac{\sin(\omega t/2)}{\omega/2}\right)^2, \qquad (12)$$

where $T = 0$.

At finite temperature, because generally a spin-1/2 system obeys Fermi statistics we can obtain the dephasing rate by multipling the to spectral weight function by $\tanh(\omega/2k_BT)$,

$$-\mathrm{Re}\ln\rho(t) = \frac{(A_{zz}\gamma_N\hbar)^2}{4}\int_{-\infty}^{\infty} d\omega \tanh(\frac{\omega}{2k_BT})\langle\{S_z(0,\omega), S_z(0,-\omega)\}\rangle\left(\frac{\sin(\omega t/2)}{\omega/2}\right)^2. \qquad (13)$$

The validity of the expression is demonstrated in by Ref. 17), and it should be required because of antiperiodic conditions (Fermi statistics). This expression is reminiscent of the Ingold-Nazarov theory in quantum tunneling events.[19)–21)] They showed the relation $P(t) = \exp K(t)$, with $K(t) = \frac{4}{\pi\hbar}\int_0^\infty \frac{J(\omega)}{\omega^2}(\coth\left(\frac{\hbar\omega}{2k_BT}\right)[(\cos(\omega t) - 1)) - i\sin\omega t])$, where $J(\omega)$ is the bath spectral density and P is a coherence function, which represents the Coulomb-blocked effect. From this correspondence, we can conclude that the bath spectral density function of the Haldane-Shastry model at finite temperature, is given by

$$J(\omega)_{HS} \equiv \frac{\pi\hbar}{8}\tanh(\omega/2k_BT)^2(A_{zz}\gamma_N\hbar)^2\langle\{S_z(0,\omega), S_z(0,-\omega)\}\rangle. \tag{14}$$

At finite temperatures, the spin bath has a smaller effect on the system, because of the possibility for saturation of the populations in the bath. A conspicuous counterexample to a spin bath is the case in which the ladder of excited states of a nonlinear bath mode narrows with increasing energy. In this case, we find enhancement of $J(\omega)$ in comparison with the spectral density of a harmonic bath. This type of temperature dependence results from the statistics of the bath.[13),15),16)] At high temperature spins of electrons behave like those of classical particles. However, in spite of this fact, such classical behavior had not been studied to this time.

Next, we examine the situation in which dephasing with dissipation occurs ($A_\perp \neq 0$). In the present case, the second-order perturbation approximation is not accurate, although for a pure dephasing event we obtain an exact result. In previous studies, Fermi's Golden Rule was used. In this case, the transverse relaxation, T_1^{-1} and the longitudinal transition rate are given by[22)] $-\ln(\eta = 1, t) = T_1^{-1} = \frac{(A_{zz}\gamma_N\hbar)^2}{4v}(3 + \frac{9}{2\pi^2}\left(\frac{\Delta}{v}\right)^2)$ and $-\ln(\xi = 1, t) = \frac{1}{2}T_1^{-1} + \frac{1}{8v}(A_\parallel\gamma_N\hbar)^2$. The results show that the dephasing suppressed as the strength of the intrachain interaction increases. This is because the fluctuations of a spinon are suppressed as J increases. Also, as Δ increases, the dephasing increases, because the spin bath behaves like as ohmic bath. It is conjectured that for a spin bath, the higher-order corrections of Δ are negative. This behavior leads us to hypothesize a finite band width of $\frac{\pi v}{2}$. In the present model, we can confirm our results, because the universality class of the Haldane-Shatry model is the Tomonaga-Luttinger liquid with $K = 1/2$,[24)] and at zero temperature, the dephasing rate of dephasing with dissipation is constant.[23)]

§5. Summary

In summary, we had examined the dephasing rate of a spin qubit system coupled with a spin chain. Due to the fluctuations of the spin of the spin chain, the dephasing is supperessed as J. The time dependence exhibits oscillatory behavior, which comes from the entangled state. We also examined dephasing with dissipation. We found that the dephasing with dissipation is suppressed as the strength of the magnetic field applied to qubit system (energy width of two-level system) is decreased. It should be noted that for pure dephasing, our results are exact and can be applied to other bath systems by using the bath spectrum density function.

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